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On the absolute N_{q_α}-summability of rth derived conjugate series

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Abstract. The object of the present paper is to study the absolute N_{q_α}-summability of rth derived conjugate series generalizing a known result.

Keywords. Fourier series; conjugate series; derived conjugate series; Nevanlinna summability; kernel.

1. Introduction

1.1.

In the year 1921, Nevanlinna [7] suggested and discussed an interesting method of summation called N_q-method. Moursund [5] applied this method for summation of Fourier series and its conjugate series. Later, Moursund [6] developed N_{q_p}-method (where p is a positive integer) and applied it to pth derived Fourier series. Samal [9] discussed N_{q_α}-method $(0 \leq \alpha < 1)$ and studied absolute N_{q_α}-summability of some series associated with Fourier series. In his Ph.D. thesis [10] he extended N_{q_p}-method of summation to N_{q_α}-method for any $\alpha \geq 0$ and studied absolute N_{q_α}-summability of Fourier series. Earlier we [8] have studied absolute N_{q_α}-summability of a series conjugate to a Fourier series. In the present paper we shall study the absolute N_{q_α}-summability of rth $(r < \alpha)$ derived series of a conjugate series.

1.2.

DEFINITION 1. [6,10]

Let $F(w)$ be a function of a continuous parameter w defined for all $w > 0$. The N_{q_α}-method consists in forming the N_{q_α}-transform or mean

$$N_{q_\alpha}F(w) = \int_0^1 q_\alpha(t)F(wt)\mathrm{d}t$$

and then considering the limit

$$\lim_{w\to\infty} N_{q_\alpha}F(w),$$

where the class of functions $q_\alpha(t)$ is such that

(1) $q_\alpha(t) \geq 0$ for $0 \leq t \leq 1$,

(2) $\int_0^1 q_\alpha(t)\mathrm{d}t = 1$,

(3) $\frac{\mathrm{d}^\beta}{\mathrm{d}t^\beta} q_\alpha(t)$ exists and is absolutely continuous for $0 \leq t \leq 1, \beta = 1,2,\dots,k-1$, where $[\alpha] = k$,

(4) $\frac{\mathrm{d}^\beta}{\mathrm{d}t^\beta} q_\alpha(t) = 0$ for $t = 1, \beta = 0,1,2,\dots,k-1$,

(5) $\frac{\mathrm{d}^k}{\mathrm{d}t^k} q_\alpha(t)$ exists for $0 < t < 1$,

(6) $(-1)^k \frac{\mathrm{d}^k}{\mathrm{d}t^k} q_\alpha(t) \geq 0$ and monotonic increasing for $0 < t < 1$,

(7) $\int_0^t \frac{Q_k(u)}{u^{1+\alpha-k}}\mathrm{d}u = O\left(\frac{Q_k(t)}{t^{\alpha-k}}\right)$,

where

$$Q_k(t) = \int_{1-t}^1 (-1)^k \frac{\mathrm{d}^k}{\mathrm{d}u^k} q_\alpha(u)\mathrm{d}u.$$

Also we set

$$Q(t) = \int_{1-t}^1 q_\alpha(u)\mathrm{d}u.$$

If $\lim_{w\to\infty} N_{q_\alpha} F(w)$ exists, we say that N_{q_α}-limit of $F(w)$ exists.

DEFINITION 2. [9,10]

Let $\sum_{n=0}^\infty u_n$ be an infinite series with $S(w) = \sum_{n\leq w} u_n$. If $\lim_{w\to\infty}\{\sum_{n\leq w} u_n Q(1-(n/w))\} = 1$, we say that $\sum u_n$ is summable by N_{q_α}-method to the sum 1. In short we write that $\sum u_n = 1(N_{q_\alpha})$. Further the series $\sum u_n$ is said to be $|N_{q_\alpha}|$-summable (absolute N_{q_α}-summable) if

$$\int_A^\infty \frac{\mathrm{d}w}{w^2}\left|\sum_{n\leq w} n u_n q_\alpha\left(\frac{n}{w}\right)\right| < \infty$$

for some positive constant A.

For $\alpha = 0$, the method reduces to the original N_q-method [7] and if α is any positive integer p, then the method reduces to N_{q_p}-method of Moursund [6].

1.3.

Let $f(t)$ be a periodic function with period 2π and Lebesgue integrable over $(-\pi,\pi)$.

Let

$$f(t) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt) \equiv \sum_{n=0}^{\infty} A_n(t). \tag{1.3.1}$$

The series conjugate to (1.3.1) at $t = x$ is given by

$$\sum_{n=1}^{\infty}(b_n \cos nx - a_n \sin nx) \equiv \sum_{n=1}^{\infty} B_n(x), \tag{1.3.2}$$

$$P(u) = \sum_{i=0}^{r-1} \frac{\theta_i}{i!}u^i \quad \text{for} \quad -\pi \le u \le \pi,$$

where θ_is for $i = 0, 1, 2, \ldots, r-1$ are arbitrary constants.

$$h(u) = \frac{\{f(x+u) - P(u)\} - (-1)^r\{f(x-u) - P(-u)\}}{2u^r},$$

$$H_0(t) = h(t),$$

$$H_\beta(t) = \frac{1}{\Gamma(\beta)}\int_0^t (t-u)^{\beta-1}h(u)\mathrm{d}u, \quad (\beta > 0),$$

$$h_\beta(t) = \Gamma(1+\beta)t^{-\beta}H_\beta(t), \quad (\beta \ge 0).$$

2. Purpose of the present paper

In the present paper we shall prove the following theorems:

Theorem 1. *Let $\beta = \alpha - r$. If $H_\beta(+0) = 0$ and $\int_0^\pi t^{-\beta}|\mathrm{d}H_\beta(t)| < \infty$, where $\beta > 0$, then the rth derived series of the conjugate series of $f(t)$ at $t = x$ is $|N_{q_\alpha}|$-summable.*

Theorem 2. *Let $\rho = \alpha - r - 1$. If $\rho \ge 0$ and $\int_0^\pi t^{-1}|h_\rho(t)|\mathrm{d}t < \infty$, then the rth derived series of the conjugate series of $f(t)$ at $t = x$ is $|N_{q_\alpha}|$-summable.*

By taking $\beta = \rho + 1, \rho \ge 0$ in Theorem 1, we can obtain Theorem 2 at once as it is known [4] that

$$H_{\rho+1}(+0) = 0 \quad \text{and} \quad \int_0^\pi t^{-\rho-1}|\mathrm{d}H_{\rho+1}(t)| < \infty$$

$$\iff h_{\rho+1}(t) \in BV(0,\pi) \quad \text{and} \quad \frac{h_{\rho+1}(t)}{t} \in L(0,\pi)$$

$$\iff \frac{h_\rho(t)}{t} \in L(0,\pi).$$

By taking $q_\alpha(t) = (\alpha+\delta)(1-t)^{\alpha+\delta-1}$, where $\delta > 0$ and $\alpha + \delta < k+1$ ($[\alpha] = k$) in Theorems 1 and 2, we obtain the following corollaries respectively.

COROLLARY 1. [3]

If $H_\beta(+0) = 0$ and $\int_0^\pi t^{-\beta}|\mathrm{d}H_\beta(t)| < \infty$, then the rth derived series of the conjugate series of $f(t)$ at $t = x$ is summable $|C, \beta + r + \delta|$, where $\beta > 0$ and $\delta > 0$.

COROLLARY 2. [3]

If $\rho \ge 0$ and $\int_0^\pi t^{-1}|h_\rho(t)|\mathrm{d}t < \infty$, then the rth derived series of the conjugate series of $f(t)$ at $t = x$ is summable $|C, \rho + r + 1 + \delta|$.

3. Notations and lemmas

3.1. *Notations*

For our purpose we use the following notations throughout this paper.

$$[\alpha] = k,$$
$$m = \min\,(k-r, r),$$
$$q^k(u) = (-1)^k \frac{\mathrm{d}^k}{\mathrm{d}u^k}\, q_\alpha(u),$$
$$(\cos\, nu)_j = \left(\frac{\mathrm{d}}{\mathrm{d}u}\right)^j \cos\, nu,$$
$$S^{i,j}(x,u) = \sum_{n\le x} (x-n)^i (\cos\, nu)_j,$$
$$G_i(w,u) = \sum_{n\le w} q_\alpha\left(\frac{n}{w}\right)\left(\frac{\mathrm{d}}{\mathrm{d}u}\right)^{k+1-i} \cos\, nu, \quad \text{for} \quad i = 0,1,2,\ldots,m,$$
$$g_i(x,w,u) = \frac{1}{k!}(-1)^k \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^k q_\alpha\left(\frac{x}{w}\right) \frac{\mathrm{d}}{\mathrm{d}x}\, S^{k,k+1-i}(x,u)$$
$$\text{for} \quad i = 0,1,2,\ldots,m.$$

3.2. *Lemmas*

We need the following lemmas for the proof of our theorem.

Lemma 1 [1]. *If $\beta > \alpha > 0, H_\alpha(t)$ is of $BV(0,\pi)$ and $H_\alpha(+0) = 0$, then $H_\beta(t)$ is an integral in $(0,\pi)$ and for almost all values of t,*

$$H'_\beta(t) = \frac{1}{\Gamma(\beta-\alpha)} \int_0^t (t-u)^{\beta-\alpha-1} \mathrm{d}H_\alpha(u).$$

Lemma 2 [6]. *If $\alpha \ge 1$, the kernel $q_\alpha(t)$ is monotonic decreasing, its derivatives of odd orders less than k are negative and monotonic increasing, its derivatives of even orders less than k are positive and monotonic decreasing and there exists a constant A_k such that*

$$\left|\frac{\mathrm{d}^\beta}{\mathrm{d}t^\beta}\, q_\alpha(t)\right| < A_k \quad (\beta = 0,1,2,\ldots,k-1)$$

and

$$\int_0^1 \left|\frac{\mathrm{d}^k}{\mathrm{d}t^k}\, q_\alpha(t)\right| \mathrm{d}t < A_k.$$

Lemma 3 [10]. *$Q_k(t)$ is continuous and monotonic increasing function of t, $Q_k(t) \ge 0, Q(0) = 0$ and $Q(1) = 1$.*

This follows directly from the definition of $Q(t)$ and $Q_k(t)$.

Lemma 4 [10,8]. *$\int_0^1 q^k(t)/((1-t)^{\alpha-k})\mathrm{d}t$ exists.*

Lemma 5 [8]. *Let $x > 0$.*

(i) *If $1/x < u \le \pi$, then*

$$S^{i,j}(x,u) = \begin{cases} O(x^i u^{-j-1}) & \text{for} \quad 0 \le j \le i, \\ O(x^j u^{-i-1}) & \text{for} \quad j > j \ge 0. \end{cases}$$

(ii) *If $1/x \ge u > 0$, then*

$$S^{i,j}(x,u) = O(x^{i+j+1}).$$

Lemma 6 [2]. *Let $\lambda = \{\lambda_n\}$ be a positive monotonic increasing sequence with $\lambda_n \to \infty$ as $n \to \infty$. Then*

$$A_\lambda(x) = A_\lambda^0(x) = \sum_{\lambda_n \le x} a_n$$

and

$$A_\lambda^r(x) = \sum_{\lambda_n \le x} (x - \lambda_n)^r a_n (r > 0).$$

If k is a positive integer,

$$A_\lambda(x) = \frac{1}{k!}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^k A_\lambda^k(x).$$

Lemma 7 [8,10]. *For $\alpha \ge 1$,*

$$\sum_{n \le w} (-1)^n n^k q_\alpha\left(\frac{n}{w}\right) = O\left\{q^k\left(1 - \frac{1}{w}\right)\right\} + O\left\{wQ_k\left(\frac{1}{w}\right)\right\}.$$

Lemma 8 [8,10]. *For $\alpha \ge 1$ and $r = 0, 1, 2, \ldots, k-1$,*

$$\sum_{n \le w} (-1)^n n^r q_\alpha\left(\frac{n}{w}\right) = O(1).$$

Lemma 9. *For $i = 0, 1, 2, \ldots, k-1$,*

$$\sum_{n \le w} (-1)^n n^i \in |Nq_\alpha|.$$

Proof. For $i = 0, 1, 2, \ldots, k-2$,

$$\begin{aligned} \int_1^\infty \frac{\mathrm{d}w}{w^2} \left| \sum_{n \le w} n(-1)^n n^i q_\alpha\left(\frac{n}{w}\right) \right| &= \int_1^\infty O(1) \frac{\mathrm{d}w}{w^2} \quad \text{by Lemma 8} \\ &= O(1) \end{aligned}$$

and

$$\int_1^\infty \frac{\mathrm{d}w}{w^2}\left|\sum_{n\le w}(-1)^n n^k q_\alpha\left(\frac{n}{w}\right)\right|$$

$$= \int_1^\infty O\left\{q^k\left(1-\frac{1}{w}\right)\right\}\frac{\mathrm{d}w}{w^2}$$

$$+ \int_1^\infty O\left\{wQ_k\left(\frac{1}{w}\right)\right\}\frac{\mathrm{d}w}{w^2} \quad \text{by Lemma 7}$$

$$= O\left(\int_0^1 q^k(u)\mathrm{d}u\right) + O\left(\int_0^1 \frac{Q_k(u)}{u}\,\mathrm{d}u\right)$$

$$= O(1)$$

by Lemma 2 and the definitions of $q^k(u)$ and $Q_k(u)$. This completes the proof of Lemma 9.

Lemma 10. *For* $i = 0,1,2,\ldots,m$,

$$G_i(w,u) = \int_1^w g_i(x,w,u)\mathrm{d}x.$$

Proof.

$$G_i(w,u) = \sum_{n\le w} q_\alpha\left(\frac{n}{w}\right)\left(\frac{\mathrm{d}}{\mathrm{d}u}\right)^{k+1-i}\cos nu$$

$$= q_\alpha(1)\sum_{n\le w}\left(\frac{\mathrm{d}}{\mathrm{d}u}\right)^{k+1-i}\cos nu$$

$$- \int_1^w \frac{\mathrm{d}}{\mathrm{d}x}\, q_\alpha\left(\frac{x}{w}\right)\left\{\sum_{n\le x}\left(\frac{\mathrm{d}}{\mathrm{d}u}\right)^{k+1-i}\cos nu\right\}\mathrm{d}x$$

$$= -\int_1^w \frac{\mathrm{d}}{\mathrm{d}x}\, q_\alpha\left(\frac{x}{w}\right)\frac{1}{k!}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^k\left\{\sum_{n\le x}(x-n)^k\left(\frac{\mathrm{d}}{\mathrm{d}u}\right)^{k+1-i}\cos nu\right\}\mathrm{d}x$$

by Lemma 6

$$= \left[\frac{1}{k!}\sum_{\rho=1}^{k-1}(-1)^\rho\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^\rho q_\alpha\left(\frac{x}{w}\right)\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{k-\rho} S^{k,k+1-i}\,(x,u)\right]_{x=1}^{w}$$

$$+ \int_1^w \frac{(-1)^k}{k!}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^k q_\alpha\left(\frac{x}{w}\right)\frac{\mathrm{d}}{\mathrm{d}x}\, S^{k,k+1-i}\,(x,u)\mathrm{d}x$$

(integrating by parts for $(k-1)$ times)

$$= \int_1^w g_i(x,w,u)\mathrm{d}x$$

as the integrated part vanishes for $x = w$ and $x = 1$.

Lemma 11. *For $wt \le \pi$ and $i = 0, 1, 2, \ldots, m$,*

$$\int_t^{t+(1/w)} u^{r-i}(u-t)^{k-\alpha} G_i(w,u)\mathrm{d}u = O(w^{\alpha-r+1}).$$

Proof. For $i = 0, 1, 2, \ldots, m$,

$$\begin{aligned}
&\int_t^{t+(1/w)} u^{r-i}(u-t)^{k-\alpha} G_i(w,u)\mathrm{d}u \\
&\quad = \int_t^{t+(1/w)} u^{r-i}(u-t)^{k-\alpha} \left(\sum_{n \le w} q_\alpha\left(\frac{n}{w}\right)\left(\frac{\mathrm{d}}{\mathrm{d}u}\right)^{k+1-i} \cos nu \right)\mathrm{d}u \\
&\quad = \int_t^{t+(1/w)} u^{r-i}(u-t)^{k-\alpha} O(w^{k+2-i})\mathrm{d}u \\
&\quad = O\left\{ \left(t+\frac{1}{w}\right)^{r-i} w^{k+2-i} \int_t^{t+(1/w)} (u-t)^{k-\alpha}\mathrm{d}u \right\} \\
&\quad = O\left\{ \left(\frac{wt+1}{w}\right)^{r-i} w^{k+2-i} \cdot \frac{1}{w^{k-\alpha+1}} \right\} \\
&\quad = O(w^{\alpha-r+1}) \quad \text{as} \quad wt \le \pi.
\end{aligned}$$

Lemma 12. *For $i = 0, 1, 2, \ldots, m$ and $wt \le \pi$,*

$$\int_{t+(1/w)}^{\pi} u^{r-i}(u-t)^{k-\alpha} G_i(w,u)\mathrm{d}u = O(w^{\alpha-r+1}).$$

Proof. By the use of Lemma 10,

$$\begin{aligned}
&\int_{t+(1/w)}^{\pi} u^{r-i}(u-t)^{k-\alpha} G_i(w,u)\mathrm{d}u \\
&\quad = \int_{t+(1/w)}^{\pi} u^{r-i}(u-t)^{k-\alpha}\mathrm{d}u \int_1^w g_i(x,w,u)\mathrm{d}x \\
&\quad = \int_{t+(1/w)}^{\pi} u^{r-i}(u-t)^{k-\alpha}\mathrm{d}u \frac{1}{k!} \\
&\qquad \times \int_1^w (-1)^k \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^k q_\alpha\left(\frac{x}{w}\right) \frac{\mathrm{d}}{\mathrm{d}x} S^{k,k+1-i}(x,u)\mathrm{d}x \\
&\quad = \frac{1}{(k-1)!} \int_1^w (-1)^k \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^k q_\alpha\left(\frac{x}{w}\right)\mathrm{d}x \\
&\qquad \times \int_{t+(1/w)}^{\pi} u^{r-i}(u-t)^{k-\alpha}\, S^{k-1,k+1-i}(x,u)\mathrm{d}u \\
&\quad = \frac{1}{(k-1)!} \int_1^w (-1)^k \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^k q_\alpha\left(\frac{x}{w}\right)\mathrm{d}x\, w^{\alpha-k} \\
&\qquad \times \int_{t+(1/w)}^{\xi} u^{r-i}\, S^{k-1,k+1-i}\,(x,u)\mathrm{d}u,
\end{aligned} \tag{3.2.1}$$

for some $t+(1/w)<\xi<\pi$, by an application of the mean value theorem. For $i\geq 2$, using Lemma 5(i) in (3.2.1), we get

$$\begin{aligned}
&\int_{t+(1/w)}^{\pi} u^{r-i}(u-t)^{k-\alpha}G_i(w,u)\mathrm{d}u\\
&\quad=\frac{1}{(k-1)!}\int_1^w(-1)^k\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^k q_\alpha\left(\frac{x}{w}\right)w^{\alpha-k}\mathrm{d}x\\
&\qquad\times\int_{t+(1/w)}^{\xi}u^{r-i}O(x^{k-1}u^{-k-2+i})\mathrm{d}u\\
&\quad=\frac{1}{(k-1)!}\int_1^w(-1)^k\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^k q_\alpha\left(\frac{x}{w}\right)w^{\alpha-k}O\left\{\frac{x^{k-1}}{\left(t+\frac{1}{w}\right)^{k-r+1}}\right\}\mathrm{d}x\\
&\quad=O\left(w^{\alpha-r+1}\int_1^w(-1)^k\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^k q_\alpha\left(\frac{x}{w}\right)x^{k-1}\mathrm{d}x\right)\\
&\quad=O\left(w^{\alpha-r+1}\int_0^1 q^k(\theta)\mathrm{d}\theta\right)\\
&\quad=O\left(w^{\alpha-r+1}\right)\quad\text{by Lemma 2.}
\end{aligned}$$

For $i=1$,

$$\begin{aligned}
&\int_{t+(1/w)}^{\xi}u^{r-i}\,S^{k-1,k+1-i}\,(x,u)\mathrm{d}u\\
&\quad=\int_{t+(1/w)}^{\xi}u^{r-1}\,S^{k-1,k}\,(x,u)\mathrm{d}u\\
&\quad=\left[u^{r-1}\,S^{k-1,k-1}\,(x,u)\right]_{t+(1/w)}^{\xi}\\
&\qquad-(r-1)\int_{t+(1/w)}^{\xi}u^{r-2}\,S^{k-1,k-1}\,(x,u)\mathrm{d}u\\
&\quad=O\left\{\frac{x^{k-1}}{\left(t+\frac{1}{w}\right)^{k-r+1}}\right\}+\int_{t+(1/w)}^{\xi}u^{r-2}O(x^{k-1}u^{-k})\mathrm{d}u\\
&\qquad\qquad\text{by Lemma 5(i)}\\
&\quad=O(w^{2k-r}).
\end{aligned}$$

Similarly, for $i=0$, integrating by parts twice and using Lemma 5(i), it follows that

$$\int_{t+(1/w)}^{\xi}u^{r-i}S^{k-1,k+1-i}(x,u)\mathrm{d}u=O(w^{2k-r}).$$

Hence, for $i \le 1$, using the above estimation in (3.2.1)

$$\begin{aligned}
&\int_{t+(1/w)}^{\pi} u^{r-i}(u-t)^{k-\alpha} G_i(w,u)\mathrm{d}u \\
&\quad = O\left(\int_1^w (-1)^k \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^k q_\alpha\left(\frac{x}{w}\right) w^{\alpha+k-r}\mathrm{d}x\right) \\
&\quad = O\left(w^{\alpha-r+1}\int_0^1 q^k(\theta)\mathrm{d}\theta\right) \\
&\quad = O(w^{\alpha-r+1}) \quad \text{by Lemma 2.}
\end{aligned}$$

This completes the proof of Lemma 12.

Lemma 13.

$$\begin{aligned}
&\int_t^{t+(1/w)} u^{r-i}(u-t)^{k-\alpha}\mathrm{d}u \int_1^{w-(\pi/t)} g_i(x,w,u)\mathrm{d}x \\
&\quad = O\left(\frac{w^{\alpha-k}}{t^{k+1-r}}\, q^k\left(1-\frac{\pi}{wt}\right)\right).
\end{aligned}$$

Proof. For some $1 < \xi < w-(\pi/t)$, by an application of the mean value theorem,

$$\begin{aligned}
&\int_t^{t+(1/w)} u^{r-i}(u-t)^{k-\alpha}\mathrm{d}u \int_1^{w-(\pi/t)} g_i(x,w,u)\mathrm{d}x \\
&\quad = \int_t^{t+(1/w)} u^{r-i}(u-t)^{k-\alpha}\mathrm{d}u \\
&\qquad \times \int_1^{w-(\pi/t)} \frac{(-1)^k}{k!}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^k q_\alpha\left(\frac{x}{w}\right)\frac{\mathrm{d}}{\mathrm{d}x}S^{k,k+1-i}(x,u)\mathrm{d}x \\
&\quad = \int_t^{t+(1/w)} \frac{1}{k!}u^{r-i}(u-t)^{k-\alpha}\left[(-1)^k\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^k q_\alpha\left(\frac{x}{w}\right)\right]_{x=w-(\pi/t)} \mathrm{d}u \\
&\qquad \times \int_\xi^{w-(\pi/t)} \frac{\mathrm{d}}{\mathrm{d}x}S^{k,k+1-i}(x,u)\mathrm{d}x \\
&\quad = \frac{1}{k!}\int_t^{t+(1/w)} \frac{u^{r-i}(u-t)^{k-\alpha}}{w^k}\, q^k\left(1-\frac{\pi}{wt}\right)\left[S^{k,k+1-i}(x,u)\right]_{x=\xi}^{w-(\pi/t)}\mathrm{d}u.
\end{aligned} \tag{3.2.2}$$

For $i=0$, using Lemma 5(i) in (3.2.2), we get

$$\begin{aligned}
&\int_t^{t+(1/w)} u^{r-i}(u-t)^{k-\alpha}\mathrm{d}u \int_1^{w-(\pi/t)} g_i(x,w,u)\mathrm{d}x \\
&\quad = \frac{1}{k!}\int_t^{t+(1/w)} \frac{u^{r}(u-t)^{k-\alpha}}{w^k}\, q^k\left(1-\frac{\pi}{wt}\right) O\left\{\frac{\left(w-\frac{\pi}{t}\right)^{k+1}}{u^{k+1}}\right\}\mathrm{d}u
\end{aligned}$$

$$= O\left(\frac{wq^k\left(1-\frac{\pi}{wt}\right)}{t^{k+1-r}}\int_t^{t+(1/w)}(u-t)^{k-\alpha}\,\mathrm{d}u\right)$$
$$= O\left(\frac{w^{\alpha-k}q^k\left(1-\frac{\pi}{kt}\right)}{t^{k+1-r}}\right).$$

For $i \geq 1$, using Lemma 5(i) in (3.2.2), we obtain

$$\int_t^{t+(1/w)} u^{r-i}(u-t)^{k-\alpha}\mathrm{d}u\int_1^{w-(\pi/t)} g_i(x,w,u)\mathrm{d}x$$
$$= \frac{1}{k!}\int_t^{t+(1/w)}\frac{u^{r-i}(u-t)^{k-\alpha}}{w^k}q^k\left(1-\frac{\pi}{wt}\right)O\left(\frac{w^k}{u^{k+2-i}}\right)\mathrm{d}u$$
$$= O\left(\frac{q^k\left(1-\frac{\pi}{wt}\right)}{t^{k+2-r}}\int_t^{t+(1/w)}(u-t)^{k-\alpha}\mathrm{d}u\right)$$
$$= O\left(\frac{w^{\alpha-k+1}}{t^{k+2-r}}q^k\left(1-\frac{\pi}{wt}\right)\right).$$
$$= O\left(\frac{w^{\alpha-k}}{t^{k+1-r}}q^k\left(1-\frac{\pi}{wt}\right)\right) \quad \text{as} \quad wt > \pi.$$

This completes the proof of Lemma 13.

Lemma 14. *For $i = 0,1,2,\ldots,m$,*

$$\int_{t+(1/w)}^{\pi} u^{r-i}(u-t)^{k-\alpha}\, S^{k,k+1-i}\left(w-\frac{\pi}{t},u\right)\mathrm{d}u = O\left(\frac{w^\alpha}{t^{k+1-r}}\right).$$

Proof. By an application of the mean value theorem for some $t+(1/w) < \xi < \pi$,

$$\int_{t+(1/w)}^{\pi} u^{r-i}(u-t)^{k-\alpha}S^{k,k+1-i}\left(w-\frac{\pi}{t},u\right)\mathrm{d}u$$
$$= w^{\alpha-k}\int_{t+(1/w)}^{\xi} u^{r-i}S^{k,k+1-i}\left(w-\frac{\pi}{t},u\right)\mathrm{d}u$$
$$= w^{\alpha-k}\left[u^{r-i}\,S^{k,k-i}\left(w-\frac{\pi}{t},u\right)\right]_{u=t+(1/w)}^{\xi}$$
$$-(r-i)w^{\alpha-k}\int_{t+(1/w)}^{\xi} u^{r-i-1}S^{k,k-i}\left(w-\frac{\pi}{t},u\right)\mathrm{d}u$$
$$= w^{\alpha-k}O\left\{\frac{w^k}{\left(t+\frac{1}{w}\right)^{k+1-r}}\right\} + w^{\alpha-k}\int_{t+(1/w)}^{\xi} u^{r-i-1}O\left(\frac{w^k}{u^{k+1-i}}\right)\mathrm{d}u$$

by Lemma 5(i)

$$= O\left(\frac{w^\alpha}{t^{k+1-r}}\right) + O\left(w^\alpha\int_{t+(1/w)}^{\xi}\frac{1}{u^{k+2-r}}\mathrm{d}u\right)$$
$$= O\left(\frac{w^\alpha}{t^{k+1-r}}\right).$$

Lemma 15. *For* $i = 0, 1, 2, \ldots, m$,

$$\int_{t+(1/w)}^{\pi} u^{r-i}(u-t)^{k-\alpha}\,\mathrm{d}u \int_{1}^{w-(\pi/t)} g_i(x,w,u)\mathrm{d}x$$
$$= O\left\{\frac{w^{\alpha-k}q^k\left(1-\frac{\pi}{wt}\right)}{t^{k+1-r}}\right\}.$$

Proof. For some $1 < \eta < w-(\pi/t)$, by an application of the mean value theorem

$$\int_{t+(1/w)}^{\pi} u^{r-i}(u-t)^{k-\alpha}\mathrm{d}u \int_{t}^{w-(\pi/t)} g_i(x,w,u)\mathrm{d}x$$
$$= \frac{1}{k!}\int_{t-(1/w)}^{\pi} u^{r-i}(u-t)^{k-\alpha}\mathrm{d}u\left[(-1)^k\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^k q_\alpha\left(\frac{x}{w}\right)\right]_{x=w-(\pi/t)}$$
$$\times \int_{\eta}^{w-(\pi/t)} \frac{\mathrm{d}}{\mathrm{d}x} S^{k,k+1-i}(x,u)\mathrm{d}x$$
$$= \frac{q^k\left(1-\frac{\pi}{wt}\right)}{k!\;w^k}\left\{\int_{t+(1/w)}^{\pi} u^{r-i}(u-t)^{k-\alpha} S^{k,k+1-i}\left(w-\frac{\pi}{t},u\right)\mathrm{d}u\right.$$
$$\left. - \int_{t+(1/w)}^{\pi} u^{r-i}(u-t)^{k-\alpha}\,S^{k,k+1-i}(\eta,u)\mathrm{d}u\right\}$$
$$= O\left(\frac{q^k\left(1-\frac{\pi}{wt}\right)w^{\alpha}}{w^k\;t^{k+1-r}}\right)$$
$$= O\left(\frac{q^k\left(1-\frac{\pi}{wt}\right)w^{\alpha-k}}{t^{k+1-r}}\right),$$

since by Lemma 14, the first integral is $O(w^{\alpha}/(t^{k+1-r}))$ and the second integral is dominated by the first integral.

Lemma 16. *For* $i = 0, 1, 2, \ldots, m$ *and* $wu > \pi$,

$$\int_{w-(\pi/t)}^{w} g_i(x,w,u)\mathrm{d}x = O\left(w^2\;u^{-k+i}Q_k\left(\frac{\pi}{wt}\right)\right).$$

Proof. For $0 \le 1$, by use of Lemma 5(i),

$$\int_{w-(\pi/t)}^{w} g_i(x,w,u)\mathrm{d}x$$
$$= \int_{w-(\pi/t)}^{w} \frac{(-1)^k}{(k-1)!}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^k q_\alpha\left(\frac{x}{w}\right)S^{k-1,k+1-i}(x,u)\mathrm{d}x$$
$$= \frac{1}{(k-1)!}\int_{w-(\pi/t)}^{w}(-1)^k\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^k q_\alpha\left(\frac{x}{w}\right)O\left(\frac{x^{k+1-i}}{u^k}\right)\mathrm{d}x$$
$$= O\left(\frac{w^{2-i}}{u^k}\int_{1-(\pi/wt)}^{1} q^k(\theta)\mathrm{d}\theta\right)$$
$$= O\left(w^2u^{-k+i}\;Q_k\left(\frac{\pi}{wt}\right)\right) \quad \text{as} \quad wu > \pi.$$

For $i \geq 2$, by use of Lemma 5(i)

$$
\begin{aligned}
&\int_{w-(\pi/t)}^{w} g_i(x,w,u)\mathrm{d}x \\
&\quad = \int_{w-(\pi/t)}^{w} \frac{(-1)^k}{(k-1)!} \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^k q_\alpha\left(\frac{x}{w}\right) S^{k-1,k+1-i}(x,u)\mathrm{d}x \\
&\quad = \frac{1}{(k-1)!} \int_{w-(\pi/t)}^{w} (-1)^k \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^k q_\alpha\left(\frac{x}{w}\right) O\left(\frac{x^{k-1}}{u^{k+2-i}}\right)\mathrm{d}x \\
&\quad = O\left(u^{-k-2+i} \int_{1-(\pi/wt)}^{1} q^k(\theta)\mathrm{d}\theta\right) \\
&\quad = O\left(w^2 u^{-k+i}\, Q_k\left(\frac{\pi}{wt}\right)\right) \quad \text{as} \quad wu > \pi.
\end{aligned}
$$

Hence

$$
\int_{w-(\pi/t)}^{w} g_i(x,w,u)\mathrm{d}x = O\left(w^2\, u^{-k+i}\, Q_k\left(\frac{\pi}{wt}\right)\right).
$$

Lemma 17. *For* $i = 0,1,2,\ldots,m$ *and* $wt > \pi$,

$$
\int_{t+(1/w)}^{\pi} u^{r-i}(u-t)^{k-\alpha} \frac{\mathrm{d}}{\mathrm{d}x} S^{k,k+1-i}(x,u)\mathrm{d}u = O\left(\frac{w^\alpha}{t^{k-r}}\right).
$$

Proof. Let $i = 0$. By mean value theorem for some $t + (1/w) < \xi < \pi$,

$$
\begin{aligned}
&\int_{t+(1/w)}^{\pi} u^{r-i}(u-t)^{k-\alpha} \frac{\mathrm{d}}{\mathrm{d}x} S^{k,k+1-i}(x,u)\,\mathrm{d}u \\
&\quad = k \int_{t+(1/w)}^{\pi} u^{r}(u-t)^{k-\alpha} S^{k-1,k+1}(x,u)\,\mathrm{d}u \\
&\quad = k w^{\alpha-k} \int_{t+(1/w)}^{\xi} u^{r} S^{k-1,k+1}(x,u)\,\mathrm{d}u \\
&\quad = k w^{\alpha-k} \left[u^r S^{k-1,k}(x,u)\right]_{u=t+(1/w)}^{\xi} \\
&\qquad - k r w^{\alpha-k} \int_{t+(1/w)}^{\xi} u^{r-1} S^{k-1,k}(x,u)\,\mathrm{d}u
\end{aligned}
$$

$$
\begin{aligned}
&= kw^{\alpha-k} O\left\{\frac{x^k}{(t+(1/w))^{k-r}}\right\} \\
&\quad - krw^{\alpha-k}\left[u^{r-1}S^{k-1,k-1}(x,u)\right]_{u=t+(1/w)}^{\xi} \\
&\quad + kr(r-1)w^{\alpha-k}\int_{t+(1/w)}^{\xi} u^{r-2}S^{k-1,k-1}(x,u)\,\mathrm{d}u \quad \text{by Lemma 5(i)} \\
&= O\left(\frac{w^\alpha}{t^{k-r}}\right) + krw^{\alpha-k} O\left\{\frac{x^{k-1}}{\left(t+\frac{1}{w}\right)^{k-r+1}}\right\} \\
&\quad + kr(r-1)w^{\alpha-k}\int_{t+(1/w)}^{\xi} u^{r-2} O\left(\frac{x^{k-1}}{u^k}\right)\,\mathrm{d}u \\
&= O\left(\frac{w^\alpha}{t^{k-r}}\right) + O\left(\frac{w^{\alpha-1}}{t^{k-r+1}}\right) \\
&= O\left(\frac{w^\alpha}{t^{k-r}}\right) \quad \text{as} \quad wt > \pi.
\end{aligned}
$$

For $i > 1$, using the technique used in the proof of Lemma 14, it can be proved that

$$
\begin{aligned}
&\int_{t+(1/w)}^{\pi} u^{r-i}(u-t)^{k-\alpha}\frac{\mathrm{d}}{\mathrm{d}x}\, S^{k,k+1-i}(x,u)\,\mathrm{d}u \\
&\quad = O\left(\frac{w^{\alpha-1}}{t^{k-r+1}}\right) \\
&\quad = O\left(\frac{w^\alpha}{t^{k-r}}\right) \quad \text{as} \quad wt > \pi.
\end{aligned}
$$

This completes the proof of Lemma 17.

Lemma 18. *For $i = 0,1,2,\ldots,m$,*

$$
\int_1^{\pi/t}\frac{\mathrm{d}w}{w^2}\left|\int_t^{\pi} u^{r-i}(u-t)^{k-\alpha}G_i(w,u)\,\mathrm{d}u\right| = O\left(\frac{1}{t^{\alpha-r}}\right).
$$

Proof.

$$
\begin{aligned}
&\int_1^{\pi/t}\frac{\mathrm{d}w}{w^2}\left|\int_t^{\pi} u^{r-i}(u-t)^{k-\alpha}G_i(w,u)\,\mathrm{d}u\right| \\
&\quad \le \int_1^{\pi/t}\frac{\mathrm{d}w}{w^2}\left|\int_t^{t+(1/w)} u^{r-i}(u-t)^{k-\alpha}G_i(w,u)\,\mathrm{d}u\right| \\
&\qquad + \int_1^{\pi/t}\frac{\mathrm{d}w}{w^2}\left|\int_{t+(1/w)}^{\pi} u^{r-i}(u-t)^{k-\alpha}G_i(w,u)\,\mathrm{d}u\right| \\
&\quad = \int_1^{\pi/t}\frac{\mathrm{d}w}{w^2}O(w^{\alpha-r+1}), \quad \text{by Lemmas 11 and 12} \\
&\quad = O\left(\frac{1}{t^{\alpha-r}}\right).
\end{aligned}
$$

Lemma 19. *For $i = 0, 1, 2, \ldots, m$ and $wt > \pi$,*

$$\int_t^{\pi} u^{r-i}(u-t)^{k-\alpha} G_i(w,u)\,\mathrm{d}u$$
$$= O\left(\frac{w^{\alpha-k} q^k \left(1-\frac{\pi}{wt}\right)}{t^{k-r+1}}\right) + O\left(\frac{w^{\alpha-k+1} Q_k\left(\frac{\pi}{wt}\right)}{t^{k-r}}\right).$$

Proof. Using Lemma 10,

$$\int_t^{\pi} u^{r-i}(u-t)^{k-\alpha} G_i(w,u)\,\mathrm{d}u$$
$$= \int_t^{\pi} u^{r-i}(u-t)^{k-\alpha}\,\mathrm{d}u \int_1^{w} g_i(x,w,u)\,\mathrm{d}x$$
$$= \int_t^{t+(1/w)} u^{r-i}(u-t)^{k-\alpha}\,\mathrm{d}u \int_1^{w} g_i(x,w,u)\,\mathrm{d}x$$
$$+ \int_{t+(1/w)}^{\pi} u^{r-i}(u-t)^{k-\alpha}\,\mathrm{d}u \int_1^{w} g_i(x,w,u)\,\mathrm{d}x$$
$$= J_1 + J_2, \quad \text{say.}$$

Using Lemmas 13 and 16,

$$J_1 = \int_t^{t+(1/w)} u^{r-i}(u-t)^{k-\alpha}\,\mathrm{d}u \int_1^{w-(\pi/t)} g_i(x,w,u)\,\mathrm{d}x$$
$$+ \int_t^{t+(1/w)} u^{r-i}(u-t)^{k-\alpha}\,\mathrm{d}u \int_{w-(\pi/t)}^{w} g_i(x,w,u)\,\mathrm{d}x$$
$$= O\left(\frac{w^{\alpha-k} q^k \left(1-\frac{\pi}{wt}\right)}{t^{k-r+1}}\right)$$
$$+ O\left(\int_t^{t+(1/w)} u^{r-k}(u-t)^{k-\alpha} w^2 Q_k\left(\frac{\pi}{wt}\right)\mathrm{d}u\right)$$
$$= O\left(\frac{w^{\alpha-k} q^k \left(1-\frac{\pi}{wt}\right)}{t^{k-r+1}}\right) + O\left(\frac{w^{\alpha-k+1} Q_k\left(\frac{\pi}{wt}\right)}{t^{k-r}}\right) \quad \text{as} \quad k \geq r$$

and

$$J_2 = \int_{t+(1/w)}^{\pi} u^{r-i}(u-t)^{k-\alpha}\,\mathrm{d}u \int_1^{w-(\pi/t)} g_i(x,w,u)\,\mathrm{d}x$$
$$+ \int_{t+(1/w)}^{\pi} u^{r-i}(u-t)^{k-\alpha}\,\mathrm{d}u \int_{w-(\pi/t)}^{w} g_i(x,w,u)\,\mathrm{d}x$$
$$= O\left(\frac{w^{\alpha-k} q^k \left(1-\frac{\pi}{wt}\right)}{t^{k-r+1}}\right) + \int_{w-(\pi/t)}^{w} \mathrm{d}x$$
$$\times \int_{t+(1/w)}^{\pi} u^{r-i}(u-t)^{k-\alpha} g_i(x,w,u)\,\mathrm{d}x \quad \text{by Lemma 15,}$$

$$= O\left(\frac{w^{\alpha-k}q^k\left(1-\frac{\pi}{wt}\right)}{t^{k-r+1}}\right) + \frac{1}{k!}\int_{w-(\pi/t)}^{w}(-1)^k\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^k q_\alpha\left(\frac{x}{w}\right)\,\mathrm{d}x$$

$$\times\int_{t+(1/w)}^{\pi}u^{r-i}(u-t)^{k-\alpha}\frac{\mathrm{d}}{\mathrm{d}x}S^{k,k+1-i}(x,u)\,\mathrm{d}x$$

$$= O\left(\frac{w^{\alpha-k}q^k\left(1-\frac{\pi}{wt}\right)}{t^{k-r+1}}\right)$$

$$+\frac{1}{k!}\int_{w-(\pi/t)}^{w}(-1)^k\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^k q_\alpha\left(\frac{x}{w}\right)O\left(\frac{w^\alpha}{t^{k-r}}\right)\mathrm{d}x \quad \text{by Lemma 17}$$

$$= O\left(\frac{w^{\alpha-k}q^k\left(1-\frac{\pi}{wt}\right)}{t^{k-r+1}}\right) + O\left(\frac{w^{\alpha-k+1}}{t^{k-r}}\int_{1-(\pi/wt)}^{1}q^k(\theta)\,\mathrm{d}\theta\right)$$

$$= O\left(\frac{w^{\alpha-k}q^k\left(1-\frac{\pi}{wt}\right)}{t^{k-r+1}}\right) + O\left(\frac{w^{\alpha-k+1}Q_k\left(\frac{\pi}{wt}\right)}{t^{k-r}}\right).$$

This completes the proof of Lemma 19.

Lemma 20. *For $i = 0,1,2,\ldots,m$,*

$$\int_{\pi/t}^{\infty}\frac{\mathrm{d}w}{w^2}\left|\int_t^\pi u^{r-i}(u-t)^{k-\alpha}G_i(w,u)\,\mathrm{d}u\right| = O\left(\frac{1}{t^{\alpha-r}}\right).$$

Proof. By the use of Lemma 19, we get

$$\int_{\pi/t}^{\infty}\frac{\mathrm{d}w}{w^2}\left|\int_t^\pi u^{r-i}(u-t)^{k-\alpha}G_i(w,u)\,\mathrm{d}u\right|$$

$$= O\left(\int_{\pi/t}^{\infty}\frac{w^{\alpha-k-2}q^k\left(1-\frac{\pi}{wt}\right)}{t^{k-r+1}}\,\mathrm{d}w\right) + O\left(\int_{\pi/t}^{\infty}\frac{w^{\alpha-k-1}Q_k\left(\frac{\pi}{wt}\right)}{t^{k-r}}\,\mathrm{d}w\right)$$

$$= O\left(\frac{1}{t^{\alpha-r}}\int_0^1\frac{q^k(\theta)}{(1-\theta)^{\alpha-k}}\,\mathrm{d}\theta\right) + O\left(\frac{1}{t^{\alpha-r}}\int_0^1\frac{Q_k(u)}{u^{\alpha-k+1}}\,\mathrm{d}u\right)$$

$$= O\left(\frac{1}{t^{\alpha-r}}\right) \quad \text{by Lemma 4.}$$

4. Proof of the theorem

Proof of Theorem 1. We have for $r \geq 1$,

$$\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^r B_n(x) = \frac{2}{\pi}\int_0^\pi\frac{(-1)^r}{2}\left\{f(x+u)-(-1)^r f(x-w)\right\}$$

$$\times\left(\frac{\mathrm{d}}{\mathrm{d}u}\right)^r\sin nu\,\mathrm{d}u$$

$$
\begin{aligned}
&= (-1)^r \frac{2}{\pi} \int_0^{\pi} h(u) u^r \left(\frac{\mathrm{d}}{\mathrm{d}u}\right)^r \sin nu \,\mathrm{d}u \\
&\quad + (-1)^r \frac{2}{\pi} \int_0^{\pi} \frac{1}{2} \{P(u) - (-1)^r P(-u)\} \left(\frac{\mathrm{d}}{\mathrm{d}u}\right)^r \sin nu \,\mathrm{d}u \\
&= \alpha_n + \beta_n, \quad \text{say.}
\end{aligned}
$$

For the proof of our theorem it is enough to show that

$$\sum \alpha_n \in |N_{q_\alpha}|$$

and

$$\sum \beta_n \in |N_{q_\alpha}|.$$

Now

$$
\begin{aligned}
n\alpha_n &= (-1)^r \frac{2}{\pi} \int_0^{\pi} n h(u) u^r \left(\frac{\mathrm{d}}{\mathrm{d}u}\right)^r \sin nu \,\mathrm{d}u \\
&= (-1)^{r+1} \frac{2}{\pi} \int_0^{\pi} h(u) u^r \left(\frac{\mathrm{d}}{\mathrm{d}u}\right)^{r+1} \cos nu \,\mathrm{d}u \\
&= (-1)^{r+1} \frac{2}{\pi} \left[\sum_{j=1}^{k-r} (-1)^{j-1} H_j(u) \left(\frac{\mathrm{d}}{\mathrm{d}u}\right)^{j-1} \right. \\
&\qquad \left. \times \left\{ u^r \left(\frac{\mathrm{d}}{\mathrm{d}u}\right)^{r+1} \cos nu \right\} \right]_{u=0}^{\pi} \\
&\qquad + (-1)^{k+1} \frac{2}{\pi} \int_0^{\pi} H_{k-r}(u) \left(\frac{\mathrm{d}}{\mathrm{d}u}\right)^{k-r} \left\{ u^r \left(\frac{\mathrm{d}}{\mathrm{d}u}\right)^{r+1} \cos nu \right\} \mathrm{d}u \\
&= J_1(n) + J_2(n), \quad \text{say.} \qquad (4.1)
\end{aligned}
$$

Since for $j = 1, 2, \ldots, k-r, H_j(+0) = O$ it is clear that $J_1(n)$ is the sum of the terms containing $(-1)^n n^p$, where p is even and $r+1 \leq p \leq k$.

By the use of Lemma 9, for $p = 1, 2, \ldots, k$,

$$\int_1^{\infty} \frac{\mathrm{d}w}{w^2} \left| \sum_{n \leq w} n^p (-1)^n q_\alpha \left(\frac{n}{w}\right) \right| < \infty.$$

Hence

$$\int_1^{\infty} \frac{\mathrm{d}w}{w^2} \left| \sum_{n \leq w} J_1(n) q_\alpha \left(\frac{n}{w}\right) \right| < \infty. \qquad (4.2)$$

Now

$$
\begin{aligned}
J_2(n) &= (-1)^{k+1}\frac{2}{\pi}\int_0^\pi H_{k-r}(u)\left(\frac{\mathrm{d}}{\mathrm{d}u}\right)^{k-r}\left\{u^r\left(\frac{\mathrm{d}}{\mathrm{d}u}\right)^{r+1}\cos nu\right\}\mathrm{d}u \\
&= \frac{2(-1)^{k+1}}{\pi\Gamma(k-\alpha+1)}\int_0^\pi\left(\frac{\mathrm{d}}{\mathrm{d}u}\right)^{k-r}\left\{u^r\left(\frac{\mathrm{d}}{\mathrm{d}u}\right)^{r+1}\cos nu\right\}\mathrm{d}u \\
&\quad\times\int_0^u(u-t)^{k-\alpha}\mathrm{d}H_\beta(t)\quad \text{by Lemma 1 as } \beta=\alpha-r \text{ and } [\alpha]=k \\
&= \frac{2(-1)^{k+1}}{\pi\Gamma(k-\alpha+1)}\int_0^\pi\mathrm{d}H_\beta(t)\int_t^\pi(u-t)^{k-\alpha}\left(\frac{\mathrm{d}}{\mathrm{d}u}\right)^{k-r} \\
&\quad\times\left\{u^r\left(\frac{\mathrm{d}}{\mathrm{d}u}\right)^{r+1}\cos nu\right\}\mathrm{d}u \\
&= \frac{2(-1)^{k+1}}{\pi\Gamma(k-\alpha+1)}\int_0^\pi\mathrm{d}H_\beta(t)\int_t^\pi(u-t)^{k-\alpha} \\
&\quad\times\left\{\sum_{i=0}^m\binom{k-r}{i}\left(\frac{\mathrm{d}}{\mathrm{d}u}\right)^i u^r\left(\frac{\mathrm{d}}{\mathrm{d}u}\right)^{k+1-i}\cos nu\right\}\mathrm{d}u \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{where } m=\min(k-r,r) \\
&= \frac{2(-1)^{k+1}}{\pi\Gamma(k-\alpha+1)}\sum_{i=0}^m\binom{k-r}{i}\frac{r!}{(r-i)!}\int_0^\pi\mathrm{d}H_\beta(t) \\
&\quad\times\int_t^\pi(u-t)^{k-\alpha}u^{r-i}\left(\frac{\mathrm{d}}{\mathrm{d}u}\right)^{k+1-i}\cos nu\,\mathrm{d}u.
\end{aligned}
$$

By the use of Lemmas 20 and 18,

$$
\begin{aligned}
&\int_1^\infty\frac{\mathrm{d}w}{w^2}\left|\sum_{n\le w}J_2(n)q_\alpha\left(\frac{n}{w}\right)\right| \\
&\quad\le\frac{2}{\pi\Gamma(k-\alpha+1)}\sum_{i=0}^m\binom{k-r}{i}\frac{r!}{(r-i)!}\int_0^\pi\left|\mathrm{d}H_\beta(t)\right|\int_1^\infty\frac{\mathrm{d}w}{w^2} \\
&\qquad\times\left|\int_t^\pi u^{r-i}(u-t)^{k-\alpha}G_i(w,u)\,\mathrm{d}u\right| \\
&\quad=\frac{2}{\pi\Gamma(k-\alpha+1)}\sum_{i=0}^m\binom{k-r}{i}\frac{r!}{(r-i)!}\int_0^\pi\left|\mathrm{d}H_\beta(t)\right| \\
&\qquad\times\left\{\int_1^{\pi/t}\frac{\mathrm{d}w}{w^2}\left|\int_t^\pi u^{r-1}(u-t)^{k-\alpha}G_i(w,u)\,\mathrm{d}u\right|\right. \\
&\qquad\left.+\int_{\pi/t}^\infty\frac{\mathrm{d}w}{w^2}\left|\int_t^\pi u^{r-i}(u-t)^{k-\alpha}G_i(w,u)\,\mathrm{d}u\right|\right\}
\end{aligned}
$$

$$= \frac{2}{\pi\Gamma(k-\alpha+1)} \sum_{i=0}^{m} \binom{k-i}{i} \frac{r!}{(r-i)!} \int_0^{\pi} \left|\mathrm{d}H_\beta(t)\right| O\left(\frac{1}{t^{\alpha-r}}\right)$$

by Lemmas 18 and 20

$$= O\left(\sum_{i=0}^{m} \binom{k-r}{i} \frac{r!}{(r-i)!} \int_0^{\pi} \frac{\left|\mathrm{d}H_\beta(t)\right|}{t^\beta}\right) \quad \text{as } \alpha - r = \beta$$

$$= O(1). \tag{4.3}$$

From (4.1), (4.2) and (4.3) it is clear that

$$\sum \alpha_n \in |N_{q_\alpha}|.$$

Let r be an odd number, i.e. $r = 2p+1$, where $p = 0,1,2,\ldots$. Then

$$\beta_n = -\frac{2}{\pi}\int_0^{\pi} \frac{1}{2}\{P(u)+P(-u)\} \left(\frac{\mathrm{d}}{\mathrm{d}u}\right)^{2p+1} \sin nu\,\mathrm{d}u$$

$$= (-1)^{p+1}\frac{2}{\pi}n^{2p+1}\int_0^{\pi}\left(\sum_{j=0}^{p}\frac{\theta_{2j}u^{2j}}{(2j)!}\right)\cos nu\,\mathrm{d}u$$

$$= (-1)^{p+1}\frac{2}{\pi}n^{2p+1}\sum_{j=0}^{p}\frac{\theta_{2j}}{(2j)!}\int_0^{u} u^{2j}\cos nu\,\mathrm{d}u$$

$$= (-1)^{p+1}\frac{2}{\pi}n^{2p+1}\sum_{j=1}^{p}\frac{\theta_{2j}}{(2j)!}(-1)^n$$

$$\times\left(\sum_{\mu=1}^{j}(-1)^{\mu+1}n^{-2\mu}\pi^{2j-2\mu+1}\frac{(2j)!}{(2j-2\mu)!}\right)$$

$$= 2(-1)^n\sum_{\mu=1}^{p}(-1)^{p+\mu}n^{2p-2\mu+1}\sum_{j=\mu}^{p}\frac{\theta_{2j}}{(2j-2\mu)!}\pi^{2j-2\mu}.$$

Let r be an even number, i.e. $r = 2p$, where $p = 1,2,\ldots$. Then

$$\beta_n = \frac{2}{\pi}\int_0^{\pi}\frac{1}{2}\{P(u)-P(-u)\}\left(\frac{\mathrm{d}}{\mathrm{d}u}\right)^{2p}\sin nu\,\mathrm{d}u$$

$$= (-1)^p\frac{2}{\pi}n^{2p}\sum_{j=1}^{p}\frac{\theta_{2j-1}}{(2j-1)!}\int_0^{\pi}u^{2j-1}\sin nu\,\mathrm{d}u$$

$$= (-1)^p\frac{2}{\pi}n^{2p}\sum_{j=1}^{p}\frac{\theta_{2j-1}}{(2j-1)!}(-1)^n$$

$$\times\left(\sum_{\mu=1}^{j}(-1)^{\mu-1}n^{-2\mu+1}\pi^{2j-2\mu+1}\frac{(2j-1)!}{(2j-2\mu)!}\right)$$

$$= 2(-1)^n\sum_{\mu=1}^{p}(-1)^{p+\mu-1}n^{2p-2\mu+1}\sum_{j=\mu}^{p}\frac{\theta_{2j-1}}{(2j-2\mu)!}\pi^{2j-2\mu}.$$

So by the use of Lemma 9,

$$\int_1^\infty \frac{\mathrm{d}w}{w^2}\left|\sum_{n\le w} n\beta_n q_\alpha\left(\frac{n}{w}\right)\right| < \infty,$$

i.e. $\sum\beta_n \in \left|N_{q_\alpha}\right|$. This terminates the proof of Theorem 1.

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